Pat O'Sullivan

Mh4714 Week 3

Week 3

0.0.1 Limits (contd.)

Terminology
If a sequence has a limit, L, then we say that it is a *convergent* sequence and that the sequence *converges to* L.

Theorem 0.1 (The Squeezing Theorem)

Let $\{a_n\}$ and $\{b_n\}$ and $\{c_n\}$ be sequences with

$$a_n \leq b_n \leq c_n \quad \forall n > T_1 \in \mathbb{R}.$$

If $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$ then $\lim_{n\to\infty} b_n = L$.

Proof

Since $\lim_{n\to\infty} a_n = L$ then for each $\epsilon > 0$ there is $T_2 \in \mathbb{R}$ such that

$$|a_n - L| < \epsilon, \quad \forall n > T_2.$$

That is

$$-\epsilon < a_n - L < \epsilon, \quad \forall n > T_2$$

which gives us

$$L - \epsilon < a_n < L + \epsilon, \quad \forall n > T_2.$$

Since $\lim_{n\to\infty} c_n = L$ then for each $\epsilon > 0$ there is $T_3 \in \mathbb{R}$ such that

$$|c_n - L| < \epsilon, \quad \forall n > T_3.$$

That is

$$-\epsilon < c_n - L < \epsilon, \quad \forall n > T_3$$

which gives us

$$L - \epsilon < c_n < L + \epsilon, \quad \forall n > T_3.$$

Let T be the maximun of $\{T_1, T_2, T_3\}$. Then we have

$$L - \epsilon < a_n \leq b_n \leq c_n < L + \epsilon, \quad \forall n > T$$

That is,

$$L - \epsilon < b_n < L + \epsilon, \quad \forall n > T$$

which gives us

$$-\epsilon < b_n - L < \epsilon, \quad \forall n > T$$

That is,

$$|b_n - L| < \epsilon, \quad \forall n > T$$

which means that

$$\lim_{n\to\infty} b_n = L.$$

$\square$

Example 0.2

The sequence $\left\{\frac{1}{n}\sin n\right\}$ is convergent:
Since

$$-1 \leq \sin(n) \leq 1 \;\; \forall n \in \mathbb{R}$$

then

$$-\frac{1}{n} \leq \frac{1}{n}\sin n \leq \frac{1}{n} \;\; \forall n > 0.$$

It is easly to show that $\lim_{n\to\infty} \frac{-1}{n} = \lim_{n\to\infty} \frac{1}{n} = 0$ and so it follows from the *Squeezing Theorem* that $\lim_{n\to\infty} \frac{1}{n}\sin n = 0.$

0.0.1.1 Sequences of the type $\{r^n\}$.

The sequences

$$\left\{\left(\frac{1}{2}\right)^n\right\} = \left\{\frac{1}{2^n}\right\} = \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16} \ldots$$

$$\left\{\left(-\frac{1}{2}\right)^n\right\} = \left\{\left(\frac{(-1)^n}{2^n}\right)\right\} = -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \frac{1}{16} \ldots$$

$$\left\{\left(\frac{3}{2}\right)^n\right\} = \left\{\frac{3^n}{2^n}\right\} = \frac{3}{2}, \frac{9}{4}, \frac{27}{8}, \frac{81}{16} \ldots$$

$$\{0.2^n\} = 0.2, 0.04, 0.008, 0.0016 \ldots$$

are all sequences of the type $\{r^n\}$.

The following lemma is useful for checking the convergence of such sequences:

Lemma 0.3

Let $k > 0$.
$(1+k)^n \geq 1 + nk$ for all integers $n \geq 1$.

Proof (By induction)

Clearly $(1+k)^1 \geq 1 + k$ and

$$\begin{aligned} (1+k)^n \geq 1 + nk \Rightarrow (1+k)^{n+1} &\geq (1+k)(1+nk) \\ &= 1 + nk + k + nk^2 \\ &= 1 + n(k+1) + nk^2 \\ &> 1 + (n+1)k \end{aligned}$$

and so it follows by induction that $(1+k)^n \geq 1 + nk$ for all integers $n \geq 1$.

$\square$

Corollary 0.4

Let $k > 0$.
$(1+k)^n \geq nk$ for all integers $n \geq 1$.

Proof

From the above lemma we have $(1+k)^n \geq 1+nk$ but clearly $1+nk \geq nk$ $\square$

Example 0.5

(i) Consider the sequence $\left\{\left(\frac{1}{3}\right)^n\right\}$.

We have
$$\left(\frac{1}{3}\right)^n = \frac{1}{3^n} = \frac{1}{(1+2)^n} \leq \frac{1}{2n}$$
because of the inequality $(1+k)^n \geq nk$ with $k=2$.

And so we have
$$0 < \left(\frac{1}{3}\right)^n \leq \frac{1}{2}\frac{1}{n}$$
and, since, $\lim_{n\to\infty} 0 = 0 = \lim_{n\to\infty} \frac{1}{n}$ it follows that $\lim_{n\to\infty} \left(\frac{1}{3}\right)^n = 0$ by the *Squeezing Theorem*.

(ii) Consider the sequence $\left\{\left(\frac{-1}{3}\right)^n\right\}$.

We have
$$-\left(\frac{1}{3}\right)^n \leq \left(\frac{-1}{3}\right)^n \leq \left(\frac{1}{3}\right)^n$$
and, since, $\lim_{n\to\infty} -\left(\frac{1}{3}\right)^n = 0 = \lim_{n\to\infty} \left(\frac{1}{3}\right)^n$ it follows that $\lim_{n\to\infty} \left(\frac{-1}{3}\right)^n = 0$ by the *Squeezing Theorem*.

(iii) Consider the sequence $\left\{\left(\frac{3}{4}\right)^n\right\}$.

We have
$$\left(\frac{3}{4}\right)^n = \frac{1}{(\frac{4}{3})^n} = \frac{1}{(1+\frac{1}{3})^n} \leq \frac{1}{\frac{1}{3}n}$$
because of the inequality $(1+k)^n \geq nk$ with $k=\frac{1}{3}$.

And so we have
$$0 < \left(\frac{3}{4}\right)^n \leq 3\frac{1}{n}$$
and, since, $\lim_{n\to\infty} 0 = 0 = \lim_{n\to\infty} \frac{1}{n}$ it follows that $\lim_{n\to\infty} \left(\frac{1}{3}\right)^n = 0$ by the *Squeezing Theorem*.

The next result is useful for helping us to prove that certain sequences are *not* convergent.

Theorem 0.6

A convergent sequence is bounded.

Proof

Let $\lim_{n\to\infty} a_n = L$ and let $\epsilon = 1$. There is $T \in \mathbb{R}$ with:

$$|a_n - L| < 1, \ \forall\ n > T$$

$$\Rightarrow L - 1 < a_n < L + 1 \ \forall\ n > T$$

Let a_{K_1} be the largest of $a_1, a_2, a_3, \ldots a_T$ (That is a_{K_1} is the largest of the first T terms of the sequence.
Let a_{K_2} be the smallest of $a_1, a_2, a_3, \ldots a_T$ (That is a_{K_2} is the smallest of the first T terms of the sequence.

Therefore if we let

$$M = \max\{a_{K_1} L + 1\}$$

and if we let

$$m = \min\{a_{K_2}, L - 1\}$$

we have

$$m \leq a_n \leq M, \ \forall\ n \in \mathbb{N}.$$

That is, the sequence is bounded. □

N.B.: One important deduction from the above theorem is that an *unbounded* sequence is **not** convergent.

Example 0.7

(i) The sequence $\{n\}$, that is, the sequence $1, 2, 3, \ldots$ is not bounded and so is not convergent.

(ii) The sequence $\{kn\}$ is not bounded for any real number $k \neq 0$.

(iii) Consider the sequence $\{3^n\}$.
We have
$$3^n = (1+2)^n \geq 2n$$
because of the inequality $(1+k)^n \geq nk$ with $k = 2$.
We can see then that $\{3^n\}$ is an unbounded sequence and therefore is not convergent.

(iv) Consider the sequence $\{(-3)^n\}$.
Since
$$(-3)^n = 3^n \geq 2n \text{ when } n \text{ is even,}$$
it follows that $\{(-3)^n\}$ is unbounded and so is not convergent.

It is clear from the above examples that

$$\{r^n\} \text{ is convergent with limit 0 when } -1 < r < 1 \text{ i.e. } |r| < 1$$

and

$\{r^n\}$ is unbounded and hence not convergent when $r < -1$ and $r > 1$ i.e. $|r| > 1$

The remaining two cases $r = 1$ and $r = -1$ have to be considered separately:

When $r = 1$ then $\{r^n\}$ is simply the constant sequence $\{1\}$ which has limit 1.
When $r = -1$ then $\{r^n\}$ is the sequence $\{(-1)^n\}$, that is, $-1, 1, -1, 1 \ldots$ which does not converge.
The sequence $\{(-1)^n\}$ is a useful example of a sequence which is bounded but *not* convergent.

0.0.1.2 Geometric Series.
The convergence properties of the geometric series $\sum\limits_{n=1}^{\infty} ar^{n-1}$ are now determined by the convergence properties of the sequence $\{r^n\}$.

Recall that a series is associated with a sequence known as the sequence of partial sums. The sequence of partial sums for a geometric series is:

$$a, a+ar, a+ar+ar^2, a+ar+ar^2+ar^3, a+ar+ar^2+ar^3+ar^4, \ldots$$

We will denote the n^{th} term of this sequence by S_n.

We can find a compact expression for S_n as follows:

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1}$$
$$\Rightarrow rS_n = ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n$$
$$\Rightarrow (1-r)S_n = a - ar^n \Rightarrow S_n = \frac{a - ar^n}{1-r} \text{ if } r \neq 1.$$

We know from above that $\lim_{n\to\infty} r^n = 0$ when $-1 < r < 1$, and does not exist when $|r| > 1$. We see then that $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a - ar^n}{1-r} = \frac{a}{1-r}$ when $-1 < r < 1$.
Therefore $\sum_{n=0}^{\infty} ar^n$ is convergent when $-1 < r < 1$ and not convergent when $|r| > 1$.

When $r = 1$ the Geometric series becomes

$$a + a + a + a \ldots.$$

That is

$$S_n = na.$$

The sequence $\{na\}$ clearly does not converge except in the trivial case where $a = 0$.